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Compensation and Responsibility: General Impossibilities and Possibilities

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Abstract

We consider a compensation problem within a responsible–sensitive egalitarian framework. We show that Arrowvian general impossibility results. However, if information about interpersonal–comparable utilities is available, this impossibility is escapable. We characterize a general class of allocation rules defined by social evaluation functions. As a corollary of our results, we obtain new characterizations of important rules such as Conditional Equality rules, Egalitarian Equivalent rules, and average versions of them.

1 Introduction

In society, individuals face different circumstances which are beyond their control, e.g., race, gender, or disability, etc. The heterogeneity of circumstances gives rise to a sometimes fatal disproportion of well–being among individuals, and hence compensating transfers are necessary to offset welfare losses due to unequal circumstances. In fact, several social security regimes are employed to protect individuals against poverty, disability, unemployment and others.

Two kinds of ethics are known to be of concern when we consider fair compensation problems. One is the *compensation principle*, which states that inequalities due to differential circumstances for which individuals are not responsible are illegitimate and should be suppressed (Fleurbaey, 2008 p.25), and the other is the *liberal reward principle*, which states that one should not advocate any "artificial" reward favoring the agents who exercise their responsibility in a particular way (Fleurbaey and Maniquet, 2005 p.5). However, as is by now well-known, these two principles are, in general, incompatible, even if we adopt an extremely weak interpretation (e.g., Fleurbaey 1994).

The aim of this paper is to investigate the general existence of compensation rules satisfying either one of these two principles under the standard framework in the literature.[1] We consider the problem

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[1]See Section 2 in Fleurbaey and Maniquet (2005), Chapter 1 and 2 in Fleurbaey (2008), and references therein.

of division of a fixed resource among the members of a society. Each individual is endowed with two kind of personal characteristics: One is *circumstance*, for which she is not responsible, and the other is *responsibility*, for which she is. Each individual's well–being is fully determined by her own characteristics and by the transfer of a one-dimensional divisible resource, say money. We assume that each individual has a quasi–linear preference. An allocation rule is a map from a characteristic profile to a feasible allocation of a resource.

In Section 2 to 4, we consider liberalist allocation rules that give priority to the liberal reward principle. We consider two criteria expressing the liberal reward principle, which are stronger than *Equal Transfer for Equal Circumstances*: Individuals with identical circumstances should be submitted to the same transfer. The first is *Strong Protection of Handicapped* (SPH): An individual unanimously considered as more handicapped than another should receive more resources. The second is *Independence of Others' Circumstances* (IOC): (i) For any pair of individuals, the order of transfer is independent from others' circumstances (provided that responsibilities are unchanged); and (ii) the order of transfer for individuals is invariant w.r.t. any permutation of circumstances (provided that responsibilities are unchanged).

In Section 3, we show an impossibility result similar to Arrow's theorem. In our setting, we can consider responsibility characteristics to be preference relations on circumstances. Suppose that a social planner can make use of only circumstances characteristics and ordinal preferences on them as an information basis. Then, any allocation rule satisfying (SPH) and (IOC) and *Independence of Irrelevant Alternatives* (IIA), that is, the order of transfer between two individuals is independent from members' preferences on other circumstances, must be *Dictatorial* in the sense that there is a dictator whose preference determines the order of transfer (Theorem 1).

As shown by Sen (1970), this Arrovian impossibility can be avoided if information about interpersonal–comparable utilities is available. An important example, in our setting, is the *Average Conditional Equality* (ACE) rule (Fleurbaey, 1995a), one of the *Conditional Equality* (CE) rules, whose reference is the average of members' responsibility characteristics. ACE rule is a nondictatorial rule satisfying (SPH), (IOC) and the cardinal version of IIA.

In Section 4, we characterize a class of f*–Conditional Equality* (fCE) rules. This class is a subclass of CE rules, and contains the ACE rule. Under an fCE rule, transfer to each individual is determined by a map f from a profile of members' numerical evaluation of her circumstance to social evaluation. We show that an allocation rule is fCE if and only if it satisfies the cardinal versions of (IOC), (IIA), and *Equal Transfer for Indifferent Circumstance* (ETIC), that is, individuals whose circumstances are unanimously considered as indifferent are compensated equally (Theorem 4). Furthermore, we characterize a class of CE rules (Theorem 2), a subclass of CE rules satisfying the cardinal version of IIA (Theorem 3), the ACE rule (Theorem 5), and CE rules with fixed reference (Theorem 6).

In Section 5, we consider egalitarian compensation rules that give priority to the compensation principle. It is well-known that there is a formal duality between liberal reward axioms and compensation axioms (e.g., Fleurbaey 2008, p.36). Thus, our results obtained in previous sections are also valid with respect to dual axioms. In particular, we provide characterizations of three important types of rules in the literature: Ψ-*Egalitarian Equivalent* rules (Theorem 8), *Average Egalitarian Equivalent* rule (Theorem 11), and *Egalitarian Equivalent* rules with fixed reference (Theorem 12).

2 Compensation Problem

2.1 The model

We consider a compensation problem in which a divisible resource, say money, is allocated among individuals with nontransferable unequal endowments. The populations is $N = \{1, \cdots, n\}$, and each individual $i \in N$ is endowed with two kinds of characteristics: $y_i \in Y$, called **circumstance**, for which she is not responsible and $z_i \in Z$, called **responsibility**, for which she is. We denote a profile of characteristics by $(y_N, z_N) = ((y_1, \cdots, y_n), (z_1, \cdots, z_n))$.

It is assumed that each individual has a quasi-linear preference. Individual i's well-being, denoted by u_i, is determined by a common function $v : Y \times Z \to \mathbb{R}$:

$$u_i(x_i, y_i, z_i) = x_i + v(y_i, z_i),$$

where $x_i \in \mathbb{R}$ is the quantity of resource transfer to which individual i is granted. As usual in the quasi-linear case, negative transfers are allowed, and for simplicity's sake the total amount to be distributed is zero.

Given any $z \in Z$, define $z(y) \equiv v(y, z)$ for each $y \in Y$. Then, we can consider $z \in Z$ to be a point in $\mathbb{R}^Y$. This immediately implies the following two facts.

Firstly, we can consider that every individual i has a preference relation (that is, a complete and transitive binary relation) R_i on Y:

$$yR_iy' \iff v(y, z_i) \geq v(y', z_i).$$

In this view, we can consider preferences on circumstances to be under the individual's responsibility, as adopted in Rawls (1971) and Dworkin (2000).

Let $\mathcal{R}$ be the set of all preference relation on Y. We denote by $\mathcal{Z}_o$, called the ordinal responsibility sets, the family of all responsibility sets such that every preference on Y is induced by some responsibility, that is,

$$\mathcal{Z}_o = \{Z | \forall R \in \mathcal{R},\ \exists z \in Z, yRy' \iff v(y, z) \geq v(y', z_i),\ \forall y, y' \in Y\}.$$

Secondly, we can define addition and scalar multiplication on Z:

$$(z + z')(y) = z(y) + z'(y) \qquad \forall z, z' \in Z,\ \forall y \in Y,$$

$$(\alpha z)(y) = \alpha z(y) \qquad \forall \alpha \in \mathbb{R},\ \forall z, \in Z,\ \forall y \in Y.$$

Let $\mathcal{Z}_c$, called the cardinal responsibility sets, be the family of all responsibility sets that are isomorphic to $\mathbb{R}^Y$. Obviously, $\mathcal{Z}_o \subset \mathcal{Z}_c$. We restrict our attention to these two families of responsibility sets $\mathcal{Z}_o$ and $\mathcal{Z}_c$.

An **economy** is denoted by $e = ((y_N, z_N)) \in Y^n \times Z^n$. We denote the set of economies by $\mathcal{E} = Y^n \times Z^n$. An **allocation** is denoted by $x_N = (x_1, \cdots, x_n) \in \mathbb{R}^n$. The set of **feasible allocations** is

$$F(e) = \left\{ x_N \in \mathbb{R}^n | \sum_{i \in N} x_i = 0 \right\}.$$

Since u_i is strictly increasing in x_i, all feasible allocations are Pareto efficient.

An **allocation rule** S is a map $e \mapsto S(e) \in F(e)$. We denote by $S_i(e)$ the transfer for individual i.

2.2 The axioms

In this subsection, we state several axioms necessary for obtaining our impossibility result.

Our first axiom states that an individual unanimously considered as more handicapped than another should receive more resources.

(SPH) Strong Protection of Handicapped :

$\forall e \in \mathcal{E}$, $\forall i, j \in N$,

$$\forall k \in N,\ v(y_i, z_k) < v(y_j, z_k) \Rightarrow S_i(e) > S_j(e).$$

This axiom is a strong version of the Protection of Handicapped axiom in Fleurbaey (1994). [2]

(PH) Protection of Handicapped (Fleurbaey 1994):

$\forall e \in \mathcal{E}$, $\forall i, j \in N$,

$$\forall k \in N,\ v(y_i, z_k) \leq v(y_j, z_k) \Rightarrow S_i(e) \geq S_j(e).$$

(SPH) sounds like a compensation axiom, however the degree of compensation can be arbitrarily low, and therefore it does not necessarily imply that the handicapped are sufficiently compensated.[3]

The next axiom is one of the *liberal reward principles.*

(IOC) Independence of Others' Circumstances:

$\forall z_N \in Z^n$, $\forall y_N, y'_N \in Y^n$ such that $y_i = y'_k$ and $y_j = y'_l$,

$$S_i(y_N, z_N) \geq S_j(y_N, z_N) \iff S_k(y'_N, z_N) \geq S_l(y'_N, z_N).$$

This axiom can be decomposed into two parts:

(i) For any pair of individuals, the order of transfer for them is independent from others' circumstances (provided that responsibilities are unchanged);

[2] More precisely, (SPH) together with (ETIC) defined below implies (PH).

[3] (PH) is really about liberal reward, and it implies (ETEC) defined below. See Fleurbaey (2008) p.37.

(ii) The order of transfer for individuals is invariant w.r.t. any permutation of circumstances (provided that responsibilities are unchanged).

One can easily check that (IOC) implies the Equal Transfer for Equal Circumstances axiom in Fleurbaey (1994).

(ETEC) Equal Transfer for Equal Circumstances (Fleurbaey, 1994):
$\forall e \in \mathcal{E}$, $\forall i, j \in N$,

$$y_i = y_j \Rightarrow S_i(e) = S_j(e).$$

The next axiom says that for any pair $\{i, j\}$ of individuals, the order of transfer depends only on the members' ordinal evaluation of i's and j's circumstances $v(y_i, z_1), \cdots, v(y_i, z_n), v(y_j, z_1), \cdots, v(y_j, z_n)$.

(IIA) Independence of Irrelevant Alternatives:
$\forall z_N, z'_N \in Z^n$, $\forall y_N \in Y^n$, $\forall i, j \in N$, if $\forall k \in N$,

$$R_k|_{y_i \cup y_j} = R'_k|_{y_i \cup y_j},$$

implies

$$S_i(y_N, z_N) \geq S_j(y_N, z_N) \iff S_i(y_N, z'_N) \geq S_j(y_N, z'_N).$$

Note that both (IOC) and (IIA) require ordinal independence. Either axiom allows order preserving change of transfer when the environment is changed to satisfy the premise of the axiom.

The next axiom states that there is a dictator in the sense that i obtains more than j if and only if the dictator considers i to be more handicapped than j.

(D) Dictatorial :
$\exists k \in N$, $\forall (y_N, z_N) \in \mathcal{E}$,

$$v(y_i, z_k) < v(y_j, z_k) \Rightarrow S_i(e) < S_j(e).$$

We say that a compensation rule S is nondictatorial, or S satisfies (ND), if S violates (D).

3 An Impossibility Result

Our first result is an impossibility theorem *à la* Arrow's theorem. Theorem 1 states that only dictatorial rules can satisfy (SPH), (IOC), and (IIA).

Theorem 1 (Impossibility Theorem (I))
Suppose that $|Y| \geq 3$ and $Z \in \mathcal{Z}_o$. Then, there exists no allocation rule satisfying (SPH), (IOC), (IIA), and (ND).

Proof:

Suppose that an allocation rule S satisfies (SPH), (IOC), and (IIA). Given e, define a binary relation $\succeq(e)$ on Y by

$$y_i \succeq (e) y_j \iff S_i(e) \leq S_j(e).$$

By convention, we denote the asymmetric (resp. symmetric) part of $\succeq(e)$ by $\succ(e)$ (resp. $\sim(e)$). Obviously, $\succeq(e)$ is transitive. By (IOC), if $y_i = y_j$, then $S_i(e) = S_j(e)$, and hence $\succeq(e)$ is reflexive. Thus, for each $e \in \mathcal{E}$, $\succeq(e)$ is a preorder.

Take arbitrary $z_N \in Z^n$. By (IOC), there exists a unique extension $\succeq(z_N)$ such that for all $y, y' \in Y$,

$$\exists y_N \in Y^n,\ y \succeq (y_N, z_N) y' \quad \Longrightarrow \quad y \succeq (z_N) y,'$$
$$\exists y_N \in Y^n,\ y \succ (y_N, z_N) y' \quad \Longrightarrow \quad y \succ (z_N) y'.$$

Suppose that z_N and z'_N represent the same profile of rankings over Y, that is, $R_k = R'_k$ for all $k \in N$. Then, by (IIA), $\succeq(z_N) = \succeq(z'_N)$. Thus, S defines the map from $F : \mathcal{R}^n \to \mathcal{R}$.

We apply Arrow's impossibility theorem to F. Since $|Y| \geq 3$ and $Z \in \mathcal{Z}_o$, the domain condition is satisfied. By (SPH),

$$\forall k \in N, y P_k y' \Rightarrow y F(R_N) y',$$

and hence weakly Paretian condition is satisfied. By (IIA), F satisfies the independence of irrelevant alternatives condition. □

Intuitively, any allocation rule satisfying (IOC) and (IIA) can be considered to be a map from $\mathcal{R}^n$ to $\mathcal{R}$. The weakly Paretian condition for this map follows from (SPH), and the independence of irrelevant alternatives condition for this map follows from (IIA). Thus, applying Arrow's theorem to this map, we can obtain the theorem.

Our impossibility result depends on (IIA). (IIA) requires that an allocation rule neglect the intensity of preferences. If allocation rules are allowed to reflect changes of intensity of preferences, a social planner can compensate for individuals by non-dictatorial rules satisfying (SPH), (IOC) and other desirable conditions.

Since in our setting agents have quasi-linear preferences, it is natural to modify (IIA) such that the difference between the transfer for i and j remains unchanged if members' cardinal evaluation of i's and j's circumstances $v(y_i, z_1), \cdots, v(y_i, z_n), v(y_j, z_1), \cdots, v(y_j, z_n)$ are unchanged (provided that circumstances are fixed).

The following is such a cardinal version of (IIA).[4]

(IIA*) Independence of Irrelevant Alternatives*:

[4] We denote by an axiom with asterisk a cardinal version of the corresponding axiom.

$\forall z_N, z'_N \in Z^n$, $\forall y_N \in Y^n$, $\forall i, j \in N$, if $\forall k \in N$,

$$v(y_i, z_k) = v(y_i, z'_k) \quad \text{and} \quad v(y_j, z_k) = v(y_j, z'_k),$$

implies

$$S_i(y_N, z_N) - S_j(y_N, z_N) = S_i(y_N, z'_N) - S_j(y_N, z'_N).$$

An important rule satisfying (IIA*) is the Average Conditional Equality rule in Fleurbaey (1995a).

Average Conditional Equality (S_{ACE}, Fleurbaey, 1995a):

$\forall e \in \mathcal{E}$,

$$\begin{aligned}(S_{ACE})_i(e) &= -\frac{1}{n}\sum_{j\in N}\left[v(y_i, z_j) - \frac{1}{n}\sum_{k\in N} v(y_k, z_j)\right] \\ &= -\frac{1}{n}\sum_{j\in N} v(y_i, z_j) + \text{Const.},\end{aligned}$$

where Const. is determined such that S_{ACE} satisfies the feasibility condition, that is, $\sum_i (S_{ACE})_i(e) = 0$. Under S_{ACE}, each agent pays the average of the evaluation of her circumstance, and this payment is equally divided among all the members.

S_{ACE} not only satisfies (IIA*) but also has several desirable properties.

Proposition 1 *S_{ACE} satisfies (PH), (SPH), (IOC), (IIA*), and (ND).*

Proof:

We have

$$(S_{ACE})_i(e) - (S_{ACE})_j(e) = \frac{1}{n}\sum_{k\in N}\left(v(y_i, z_k) - v(y_j, z_k)\right).$$

□

4 f-Conditional Equality Rules

4.1 Φ-Conditional Equality Rules

We begin with Φ-Conditional Equality rules in Fleurbaey (1995a).

Φ-Conditional Equality ($S_{\Phi CE}$, Fleurbaey 1995a):

There exists $\Phi : Z^n \to Z$, $\forall e \in \mathcal{E}$,

$$(S_{\Phi CE})_i(e) = -v(y_i, \Phi(z_N)) + \frac{1}{n}\sum_{j\in N} v(y_j, \Phi(z_N)).$$

We denote the set of all Φ-conditional equality rules by $\mathcal{S}_{\Phi CE}$. The average conditional equality rule is a Φ-CE such that $v(y, \Phi(z_N)) = \frac{1}{n}\sum_j v(y, z_j)$.

To characterize Φ-CE, we need the following cardinal version of (IOC*).

(IOC*) Independence of Others' Circumstances:

$\forall z_N \in Z^n$, $\forall y_N, y'_N \in Y^n$ such that $y_i = y'_k$ and $y_j = y'_l$,

$$S_i(y_N, z_N) - S_j(y_N, z_N) = S_k(y'_N, z_N) - S_l(y'_N, z_N).$$

Obviously, (IOC*) implies (IOC), but the converse is not true.

Theorem 2 (Characterization of ΦCE)

Suppose that $Z \in \mathcal{Z}_c$. Then, $S \in \mathcal{S}_{\Phi CE}$ if and only if S satisfies (IOC*).

Proof:

(i) Necessity: Obvious.

(ii) Sufficiency: At first, we show the following lemma.

Lemma 1 *Let A be an arbitrary set and $h : A^2 \to \mathbb{R}$. If h satisfies the triangular equality, that is, for each $a, b, c \in A$,*

$$h(a, c) = h(a, b) + h(b, c),$$

then there exists $g : A \to \mathbb{R}$ such that

$$h(a, b) = g(a) - g(b), \ \forall a, b \in A. \tag{1}$$

Furthermore, another function $g' : A \to \mathbb{R}$ also satisfies Eq. (1) if and only if $g' = g + c$ for some $c \in \mathbb{R}$.

Proof:

The triangular equality implies that

$$\begin{aligned} h(a, a) &= h(a, a) + h(a, a) = 0, \\ h(a, b) &= h(a, a) - h(b, a) = -h(b, a). \end{aligned}$$

Take an arbitrary $a_0 \in A$ and define $g : A \to \mathbb{R}$ by

$$g(a) \equiv h(a, a_0), \ \forall a \in A.$$

Then, we have

$$\begin{aligned} h(a, b) &= h(a, a_0) + h(a_0, b) \\ &= h(a, a_0) - h(b, a_0) \\ &= g(a) - g(b). \end{aligned}$$

Suppose that $g' : A \to \mathbb{R}$ also satisfies Eq. (1). Then,

$$g'(a) - g(a) = h(a,b) + g'(b) - (h(a,b) + g(b)) = g'(b) - g(b)$$

Therefore, there exists $c \in \mathbb{R}$ such that $g' = g + c$. Conversely, if $g' = g + c$, then g' satisfies Eq. (1). □

Define $h : Y^2 \times Z^n \to \mathbb{R}$ by

$$h(y, y'|z_N) = S_i(e) - S_j(e),$$

where $e = (y_N, z_N)$ such that $\exists i, j \in N$, $y_i = y$ and $y_j = y'$. By (IOC*), h is well-defined. Obviously,

$$h(y, y'|z_N) = h(y, y''|z_N) + h(y'', y'|z_N), \quad \forall y, y', y'' \in Y,\ \forall z_N \in Z^n.$$

Thus, by Lemma 1, there exists a function $g : Y \times Z^n \to \mathbb{R}$ such that

$$S_i(e) - S_j(e) = g(y_i|z_N) - g(y_j|z_N).$$

By the feasibility condition, we must have $\sum_i S_i(e) = 0$. Consequently,

$$S_i(e) = g(y_i|z_N) + \frac{1}{n}\sum_j g(y_j|z_N).$$

Since $Z \in \mathcal{Z}_c$, Z is isomorphic to $\mathbb{R}^Y$. Define $\Phi : Z^n \to Z$ by

$$\Phi(z_N)(y) = -g(y|z_N),$$

which completes the proof. □

4.2 Separable Conditional Equality Rules

We consider here a subclass of $S_{\Phi CE}$ that satisfies (IIA*).

Separable Conditional Equality (S_{SCE}):

There exists $\Phi : Z^n \to Z$, $\forall e \in \mathcal{E}$,

$$(S_{\Phi CE})_i(e) = -v(y_i, \Phi(z_N)) + \frac{1}{n}\sum_{j \in N} v(y_j, \Phi(z_N)),$$

where $\Phi(z_N)(y)$ depends only on $z_1(y), \cdots, z_n(y)$. We denote the set of all separable conditional equality rules by $\mathcal{S}_{SCE}$.

Theorem 3 (Characterization of SCE)

Suppose that $Z \in \mathcal{Z}_c$. Then, $S \in \mathcal{S}_{SCE}$ if and only if S satisfies (IOC*) and (IIA*).

Proof:

Obvious. □

4.3 f-Conditional Equality Rules

We consider here a subclass of S_{SCE} such that i's transfer is determined by a social evaluation function f which is a map to a profile of individuals' evaluation for circumstances to a social evaluation for circumstances.

f-Conditional Equality (S_{fCE}):

There exists $f : \mathbb{R}^n \to \mathbb{R}$, $\forall e \in \mathcal{E}$,

$$(S_{fCE})_i(e) = f(v(y_i, z_1), \cdots, v(y_i, z_N)) - \frac{1}{n} \sum_{j \in N} f(v(y_j, z_1), \cdots, v(y_j, z_n)).$$

We denote the set of all f-conditional equality rules by $\mathcal{S}_{fCE}$.

It is desirable (in the sense of the liberal reward principle) that individuals whose circumstances are unanimously considered as indifferent be compensated equally.

(ETIC) Equal Transfer for Indifferent Circumstances:

$\forall e \in \mathcal{E}$, $\forall i, j \in N$,

$$\forall k \in N,\ v(y_i, z_k) = v(y_j, z_k) \Rightarrow S_i(e) = S_j(e).$$

(ETIC) implies (ETEC), and (ETIC) is implied by (PH).

Theorem 4 (Characterization of fCE)
$S \in \mathcal{S}_{fCE}$ if and only if S satisfies (ETIC), (IOC), and (IIA*).*

Proof:

(i) Necessity: Suppose that $S \in S_{fCE}$. Then, for each $e = (y_N, z_N) \in \mathcal{E}$,

$$S_i(e) - S_j(e) = f(v(y_i, z_1), \cdots, v(y_i, z_N)) - f(v(y_j, z_1), \cdots, v(y_j, z_N)).$$

Therefore, S satisfies (ETIC), (IOC*), and (IIA*).

(ii) Sufficiency: Suppose that S satisfies (ETIC), (IOC*), and (IIA*). By Theorem 3, there exists $\Phi : Z^n \to Z$, for each $e \in \mathcal{E}$,

$$S_i(e) = -v(y_i, \Phi(z_N)) + \frac{1}{n} \sum_{j \in N} v(y_j, \Phi(z_N)).$$

Furthermore, by (ETIC) and (IIA*), for each $e \in \mathcal{E}$,

$$\forall k \in N, z_k(y_i) = z_k(y_j) \Rightarrow \Phi(z_N)(y_i) = \Phi(z_N)(y_j),$$

which completes the proof. □.

4.4 Additional Properties

In this subsection, we consider several additional properties that fCE rules may or may not satisfy.

(An) Anonymity:

$\forall e = (y_N, z_N) \in \mathcal{E}, \forall \pi \in \Pi_N$,

$$S_{\pi(N)}(e) = S(y_{\pi(N)}, z_{\pi(N)}),$$

where Π_N denotes the set of all permutations over n-elements.

(PR) Positively Responsive

$\forall y_N \in Y^n$, $\forall i \in N$, $\forall z_N, z'_N \in Z^n$, if $\exists k \in N$

$$\begin{aligned} v(y_i, z_k) &> v(y_i, z'_k), \\ v(y_j, z_k) &= v(y_j, z'_k) \quad \forall j \neq k, \\ z_j &= z'_j \quad \forall j \neq k, \end{aligned}$$

then

$$S_i(y_N, z_N) < S_i(y_N, z_N).$$

(NR) Nonnegatively Responsive

$\forall y_N \in Y^n$, $\forall i \in N$, $\forall z_N, z'_N \in Z^n$, if $\exists k \in N$

$$\begin{aligned} v(y_i, z_k) &> v(y_i, z'_k), \\ v(y_j, z_k) &= v(y_j, z'_k) \quad \forall j \neq k, \\ z_j &= z'_j \quad \forall j \neq k, \end{aligned}$$

then

$$S_i(y_N, z_N) \leq S_i(y_N, z_N).$$

(Ad) Additivity:

$\forall y_n \in Y^n$, $\forall z_N, z'_N \in Z^n$,

$$S_i(y_N, z_N + z'_N) - S_j(y_N, z_N + z'_N) = S_i(y_N, z_N) - S_j(y_N, z_N) + S_i(y_N, z'_N) - S_j(y_N, z'_N).$$

The next axiom expresses the compensation principle. It states that if all individuals have the same responsibility characteristics, then they obtain the same well-being.

(EWBUR) Equal Well-Being for Uniform Responsibility (Fleurbaey 1994, Bossert 1995):

$\forall e = (y_N, z_N) \in \mathcal{E}$ such that $z_k = z$, $\forall k \in N$,

$$\forall i, j \in N,\ u_i(e|S) = u_j(e|S).$$

This axiom is the strongest compensation axiom in the literature which is compatible with (ETEC). As we will show below, there are many fCE rules satisfying (EWBUR).

Proposition 2 *Suppose that $S \in \mathcal{S}_{fCE}$. Then,*

(i) S satisfies (An) if and only if f is symmetric.

(ii) S satisfies (D) if and only if f depends only on $v(\cdot, z_k)$ for some $k \in N$.

(iii) S satisfies (SPH) if and only if f is weakly decreasing, that is, for all $a, b \in \mathbb{R}^n$, $a \gg b$ implies $f(a) < f(b)$.

(iv) S satisfies (PR) if and only if f is strictly decreasing, that is, for all $a, b \in \mathbb{R}^n$, $a \geq b$ and $a \neq b$ implies $f(a) < f(b)$.

(v) S satisfies (NR) if and only if f is nonincreasing, that is, for all $a, b \in \mathbb{R}^n$, $a \geq b$ implies $f(a) \leq f(b)$.

(vi) S satisfies (Ad) if and only if f is additive, that is, $f(a + b) = f(a) + f(b)$ $\forall a, b \in \mathbb{R}^n$.

(vii) S satisfies (EWBUR) if and only if $f(\alpha, \cdots, \alpha) = -\alpha$ $\forall \alpha \in \mathbb{R}$.

4.5 Average Conditional Equality

We now characterize the ACE rule. Note that ACE is an fCE rule such that f is the arithmetic mean of carguments. The following lemma provides the necessary and sufficient condition for a function being the arithmetic mean.

Lemma 2 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is symmetric and satisfies*

$$f(a + b) = f(a) + f(b) \qquad \forall a, b \in \mathbb{R}^n,$$

$$f(\alpha, \cdots, \alpha) = -\alpha \qquad \forall \alpha \in \mathbb{R}.$$

Then, and only then

$$f(\alpha_1, \alpha_2, \cdots, \alpha_n) = \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n}.$$

Proof:

See Aczél (2006) Theorem 4 p. 239. □

Theorem 5 (Characterization of ACE)

Suppose that $S \in \mathcal{S}_{GCE}$. $S = S_{ACE}$ if and only if S satisfies (An), (Ad) and (EWBUR).

Proof:

□

Corollary 1 *$S = S_{ACE}$ if and only if satisfies (ETIC), (IOC*), (IIA*), (An), (Ad) and (EWBUR).*

4.6 Conditional Equality with Fixed Reference

Conditional Equality with Fixed Reference (S_{CE}, Fleurbaey, 1995a):

$\forall e \in \mathcal{E}$,

$$(S_{CE})_i(e) = -v(y_i, \tilde{z}) + \frac{1}{n} \sum_{j \in N} v(y_j, \tilde{z}),$$

where $\tilde{z} \in Z$.

We denote by $\mathcal{S}_{CE}$ the set of all conditional equality rule with fixed reference.

(IRC) Independence of Responsibility Characteristics (Bossert, 1995):

$\forall y_N \in Y^n$, $\forall z_N, z'_N \in Z^n$,

$$S(y_N, z_N) = S(y_N, z'_N).$$

Theorem 6 (Characterization of CE)
$S \in \mathcal{S}_{CE}$ if and only if S satisfies (IRC) and (IOC).*

Proof:

(i) Necessity: Obvious.

(ii) Sufficiency: Suppose that S satisfies (IRC) and (IOC*). By Theorem 2, $S \in \mathcal{S}_{\Phi CE}$. Furthermore, (IRC) implies that Φ is a constant map. □

5 Duality

It is well-known that there is a formal duality between liberal reward axioms relating y_i to x_i and compensation axioms relating z_i to u_i.

5.1 The Axioms

For notational simplicity, let $u_i(e|S) = S_i(e) + v(y_i, z_i)$.

(SAM) Strong Acknowledged Merit :

$\forall e \in \mathcal{E}$, $\forall i, j \in N$,

$$\forall k \in N,\ v(y_k, z_i) > v(y_k, z_j) \Rightarrow u_i(e|S) > u_j(e|S).$$

(SAM) is a dual of (SPH). The following *Acknowledged Merit* is a dual of (PH).

(AM) Acknowledged Merit (Fleurbaey 2008):

$\forall e \in \mathcal{E}$, $\forall i, j \in N$,

$$\forall k \in N,\ v(y_k, z_i) \geq v(y_k, z_j) \Rightarrow u_i(e|S) \geq u_j(e|S).$$

We now state the dual of (IOC) and (IOC*).

(IOR) Independence of Others' Responsibility:

$\forall y_N \in Y^n$, $\forall z_N, z'_N \in Z^n$ such that $z_i = z'_k$ and $z_j = z'_l$,

$$u_i((y_N, z_N)|S) \geq u_j((y_N, z_N)|S) = u_k((y_N, z'_N)|S) \geq u_l((y_N, z'_N)|S).$$

(IOR*) Independence of Others' Responsibility*:

$\forall y_N \in Y^n$, $\forall z_N, z'_N \in Z^n$ such that $z_i = z'_k$ and $z_j = z'_l$,

$$u_i((y_N, z_N)|S) - u_j((y_N, z_N)|S) = u_k((y_N, z'_N)|S) - u_l((y_N, z'_N)|S).$$

(IOR*) implies (IOR).

The next axiom is the dual of (ETEC), which states that individuals with identical responsibility characteristics should have the same level of well-being.

(EWBER) Equal Well-Being for Equal Responsibility (Fleurbaey, 1994):

$\forall e \in \mathcal{E}$, $\forall i, j \in N$,

$$z_i = z_j \Rightarrow u_i(e|S) = u_j(e|S).$$

(EWBER) is implied by (IOR).

(EWBIR) :

$\forall e \in \mathcal{E}$, $\forall i, j \in N$,

$$\forall k \in N, v(z_i, y_k) = v(z_j, y_k) \Rightarrow u_i(e|S) = u_j(e|S).$$

(EWBIR) is implied by (EWBER).

Given any y_N, let $\tilde{R}_i$ be the preference relation on Z defined by $z\tilde{R}_i z' \iff y_i(z) \geq y_i(z')$, where $y_i(z) \equiv v(y_i, z)$. By regarding y as a function from Z to $\mathbb{R}$, we can consider circumstances to be the capability of agents to transform resources into well-being, this view is adopted in Moreno-Ternero and Roemer (2005). We define $\mathcal{Y}_o$ ($\mathcal{Y}_c$) as similar to Z_o ($\mathcal{Z}_c$). The dual of (IIA) and (IIA*) are as follows.

(IIR) Independence of Irrelevant Responsibility:

$\forall y_N, y'_N \in Y^n$, $\forall z_N \in Z^n$, $\forall i, j \in N$, if $\forall k \in N$,

$$\tilde{R}_k|_{z_i \cup z_j} = \tilde{R}'_k|_{z_i \cup z_j}$$

implies

$$u_i((y_N, z_N)|S) \geq u_j((y_N, z_N)|S) \iff u_i((y_N, z'_N)|S) \geq u_j((y_N, z'_N)|S).$$

(IIR*) Independence of Irrelevant Responsibility*:

$\forall y_N, y'_N \in Y^n$, $\forall z_N \in Z^n$, $\forall i, j \in N$, if $\forall k \in N$,

$$v(y_k, z_i) = v(y_k, z'_i) \quad \text{and} \quad v(y_k, z_j) = v(y_k, z'_j),$$

implies

$$u_i((y_N, z_N)|S) - u_j((y_N, z_N)|S) = u_i((y_N, z'_N)|S) - u_j((y_N, z'_N)|S).$$

(D') Dictatorial' :

$\exists k \in N$, $\forall (y_N, z_N) \in \mathcal{E}$,

$$u_i(e|S) \geq u_j(e|S) \iff v(y_k, z_i) \leq v(y_k, z_i).$$

The following axiom representing the liberal reward principle is the dual of (EWBUR).

(ETUC) :

$\forall e = (y_N, z_N) \in \mathcal{E}$ such that $y_k = z$, $\forall k \in N$,

$$\forall i, j \in N, S_i(e) = S_j(e).$$

The dual of (IRC) is the following axiom.

(AS) Additive Solidarity (Bossert, 1995):

$\forall y_N, y'_N \in Y^n$, $\forall z_N \in Z^n$, $\forall i, j \in N$,

$$u_i((y_N, z_N)|S) - u_i((y'_N, z_N)|S) = u_j((y_N, z_N)|S) - u_j((y'_N, z_N)|S).$$

5.2 Impossibility Theorem

As a dual of Theorem 1, we obtain the following impossibility theorem.

Theorem 7 (Impossibility Theorem (II))

Suppose that $|Y| \geq 3$ and $Y \in \mathcal{Y}_o$. Then, there exists no allocation rule satisfying (SAM), (IOR), (IIR), and (ND').

5.3 Ψ-Egalitarian Equivalent Rules

We now characterize the Ψ-Egalitarian Equivalent Rules in Fleurbaey (1995a).

Ψ-Egalitarian-Equivalent ($S_{\Psi EE}$, Fleurbaey 1995a):

There exists $\Psi : Y^n \to Y$, $\forall e \in \mathcal{E}$,

$$(S_{\Psi EE})_i(e) = -v(y_i, z_i) + v(\Phi(y_N), z_i) + \frac{1}{n} \sum_{j \in N} v(\Psi(y_N), z_j).$$

We denote the set of all Ψ-egalitarian equivalent rules by $\mathcal{S}_{\Psi EE}$.

Theorem 8 (Characterization of Ψ-EE)

Suppose that $Y \in \mathcal{Y}_c$. Then, $S \in \mathcal{S}_{\Psi EE}$ if and only if S satisfies (IOR).*

5.4 Separable Egalitarian Equivalent Rules

We consider here the subclass of $S_{\Psi EE}$ which satisfies (IIR*).

Separable Egalitarian-Equivalent (S_{SEE}, Fleurbaey 1995a):

There exists $\Psi : Y^n \to Y$, $\forall e \in \mathcal{E}$,

$$(S_{\Psi EE})_i(e) = -v(y_i, z_i) + v(\Phi(y_N), z_i) + \frac{1}{n} \sum_{j \in N} v(\Psi(y_N), z_j),$$

where $\Psi(y_N)(z)$ depends only on $y_1(z), \cdots, y_n(z)$. We denote the set of all Ψ-egalitarian equivalent rules by $\mathcal{S}_{SEE}$.

Theorem 9 (Characterization of SEE)

Suppose that $Y \in \mathcal{Y}_c$. Then, $S \in \mathcal{S}_{SEE}$ if and only if S satisfies (IOR) and (IIR*).*

5.5 f-Egalitarian Equivalent Rules

We now consider the dual of f-conditional equivalent rules.

f-Egalitarian Equivalent (S_{fEE}):

There exists $f : \mathbb{R}^n \to \mathbb{R}$, $\forall e \in \mathcal{E}$,

$$(S_{fCE})_i(e) = -v(y_i, z_i) + f(v(y_1, z_i), \cdots, v(y_n, z_i)) - \frac{1}{n} \sum_{j \in N} \big(f(v(y_1, z_j), \cdots, v(y_n, z_j)) - v_j(y_j, z_j)\big).$$

We denote the set of all f-egalitarian equivalent rules by $\mathcal{S}_{fEE}$.

Theorem 10 (Characterization of fEE)

$S \in \mathcal{S}_{fEE}$ if and only if S satisfies (EWBER), (IOR), and (IIR*).*

5.6 Additional Properties

As in the case of fCE rules, we can consider several additional properties of fEE rules by modifying the axioms defined in Section 4.4 in an appropriate way. For instance, the additivity axiom is modified to

(Ad') Additivity':

$\forall z_n \in Z^n$, $\forall y_N, y'_N \in Y^n$,

$$u_i(y_N + y'_N, z_N|S) - u_j(y_N + y'_N, z_N|S) = u_i(y_N, z_N|S) - u_j(y_N, z_N|S) + u_i(y'_N, z_N|S) - u_j(y'_N, z_N|S).$$

5.7 Average Egalitarian-Equivalent Rules

We here characterize the Average Egalitarian Equivalent rule in Moulin (1994).

Average Egalitarian Equivalent (S_{AEE}, Moulin, 1994):

$\forall e \in \mathcal{E}$,

$$(S_{AEE})_i(e) = -v(y_i, z_i) + \frac{1}{n}\sum_{j\in N}\left[v(y_j, z_j) - \frac{1}{n}\sum_{k\in N}\big(v(y_j, z_j) - v(y_k, z_j)\big)\right].$$

Theorem 11 (Characterization of AEE)

Suppose that $S \in \mathcal{S}_{fEE}$. Then, $S = S_AEE$ if and only if S satisfies (An), (Ad') and (ETUC).

5.8 Egalitarian-Equivalent with Fixed Reference

Egalitarian-Equivalent with Fixed Reference (S_{EE}, Fleurbaey, 1995a):

$\forall e \in \mathcal{E}$,

$$(S_{EE})_i(e) = -v(y_i, z_i) + v(\tilde{y}, z_i) + \frac{1}{n}\sum_{j\in N}\left(v(y_j, z_j) - v(\tilde{y}, z_j)\right),$$

where $\tilde{y} \in Y$.

We denote by $\mathcal{S}_{EE}$ the set of all egalitarian-equivalent rules with fixed reference.

Theorem 12 (Characterization of EE)

$S \in \mathcal{S}_{EE}$ if and only if S satisfies (AS) and (IOR*).

6 Conclusions

We have analyzed a compensation problem within a responsible–sensitive egalitarian framework. Firstly, we have shown that Arrowvian general impossibility results. However, if information about interpersonal–comparable utilities is available, this impossibility is escapable. We have characterized a general class of allocation rules defined by social evaluation functions. As a corollary of our results, we have obtained new characterization of important rules such as Conditional Equality rules, Egalitarian Equivalent rules, and the average versions of them.

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